Pareto's Mechanical Dream[1]

Franco Donzelli
Università degli Studi di Milano
e-mail: franco.donzelli@unimi.it

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1 Introduction

Vilfredo Pareto, trained as a mechanical engineer during his university studies, then a managing director of business firms for a few years, turned relatively late in his life to both the academic profession and the systematic study of economics. However, after embracing his new career in the early 1890s, he devoted twenty years of unremitting efforts (essentially between1892 and 1911) to the development of the economic science, quickly becoming one of the leading figures in that area of economic research, referred to by Pareto as "mathematical economics", that was then practiced by a small, though rapidly growing, minority of scholars. As is well-known, as far as mathematical economics is concerned, Pareto's main purpose and achievement was to perfect and expand General Equilibrium Theory, that is, the theory that Pareto had inherited from Walras and that he was inclined to regard as the core of "pure economics". In this respect, while in his early works, particularly in his first long theoretical essay, published in Italian in five instalments in 1892-93, as well as in his *Cours d'économie politique,* first published in two volumes in 1896-97, Pareto confined himself to a brilliant restatement of Walras' original theory, with only a few ingenious embellishments, in his later writings he strove to turn General Equilibrium Theory into a more and more abstract and general theory of rational choice and social interaction: indeed, this ambitious objective is made explicit in the mathematical appendix to the French edition of Pareto's *Manuel d'économie politique*, published in 1909, as well as in his French encyclopaedia article, "L'économie mathématique", published in 1911, which may be regarded as his last significant contribution to economic theory strictly speaking. (As is well-known, in fact, after 1911 Pareto almost exclusively addressed his research interests towards sociology and related topics.)

Now, while Pareto's perspective and purposes broadened and partly changed over the two decades he consecrated to economics, he always kept faithful to one methodological principle which in fact constantly recurs, albeit with varying emphasis, in all of his economic writings of that period: namely, the principle that the science of economics has much to learn from the science of mechanics or, more precisely, that rational mechanics ought to be taken as the model after which to shape theoretical economics. In a sense, this idea of Pareto's was nothing original in the economics community of that period; rather, it was almost a commonplace among the economists who chiefly contributed to the foundation and early consolidation of the approach which would later be called "marginalist"or "neoclassical economics". As a matter of fact, and just to give a few examples, two founding fathers of the approach out of the conventional three, namely, Jevons (1871) and Walras (1874-77), made systematic use of the mechanical analogy in developing and justifying their new theories (towards the end of his life, Walras (1909) will come back to the idea of a fundamental methodological similarity between mechanics and economics, by publishing an essay with the revealing title "Économique et mécanique"); a few years after the fatal 1871, Edgeworth (1881) frequently resorted to his knowledge of classical mechanics to strengthen his economic arguments, whereas Marshall (1890), his notorious predilection for biological analogies notwithstanding, did not refrain from filling up his *magnum opus* with mechanical illustrations and digressions; finally, in the very year in which Pareto entered the scientific arena with his first important contribution to economics, Fisher (1892) published his famous dissertation, where a general equilibrium model is discussed which is explicitly constructed as a model of hydromechanical equilibrium. Yet, even if most of his contemporaries apparently shared in common the idea that the newly born approach of mathematical economics should draw its inspiration from the extraordinary results reached over the centuries by the queen of the natural sciences, that is, classical mechanics, it was Pareto who more than any other economist of that period tried to turn that somewhat vague methodological prescription into a precise analytical program.

The lines of such program are reiterated over and over again in practically all of Pareto's writings concerning theoretical economics (see, in particular, Pareto (1892-93), (1896-97), (1900), (1901a), (1901b), (1902), (1906), (1909), (1911)). Yet the work where Pareto's stance is most explicitly stated is undoutedly the *Cours*, especially Vol. II. Here, in a famous passage, Pareto (1896-97, Vol. II, pp. 12-13) summarizes his point of view on the relationship between mechanics and economics in the following way:

592. Il serait inutile d'insister sur ce sujet si l'Economie politique n'était étudié que par des personnes ayant des connaissances de mécanique rationelle. L'équilibre d'un système économique présente des analogies frappantes avec l'équilibre d'un système mécanique[1] [Pareto's argument continues in following footnote]

(**592**)[1] Il n'est peut-être pas inutile de présenter un tableau des analogies qui existent entre le phénomène mécanique et le phénomène sociale. [...]

Phénomène mécanique	**Phénomène sociale**
Un certain nombre de corps matériels étant donnés, on étudie les rapports d'équilibre et de mouvement qu'ils peuvent avoir entre eux, en faisant abstraction des autres propriétés. On a ainsi une étude de *mécanique.*	Une société étant donné, on étudie les rapports que la production et l'échange de la richesse suscitent entre les hommes, en faisant abstraction des autres circonstances. On a ainsi une étude d'*économie politique.*
Cette science de la mécanique se divise elle-même entre deux autres. On considère des points matériels et des lignes inextensibles. On a ainsi une science pure: la mécanique rationelle, qui étudie d'une manière abstraite l'équilibre des forces et le mouvement. La partie la plus facile est la science de l'équilibre. Le principe de d'Alembert, en considérant les forces d'inertie, permet de réduire la dynamique à un problème de statique.	Cette science de l'économie politique se divise elle-même entre deux autres. On considère l'*homo oeconomicus*, n'agissant qu'en vertu des forces économiques. On a ainsi l'économie politique pure, qui étudie, d'une manière abstraite, les manifestations de l'ophelimité. La seule partie que nous commençons à bien connaître est la partie qui traite de l'équilibre. Nous avons pour les systèmes économiques un principe semblable à celui de d'Alembert (586[1]); mais nos connaissances sur ce sujet sont encore des plus imparfaites. La théorie des crises économiques fournit pourtant un exemple d'étude de dynamique économique.

A la mécanique rationelle fait suite la mécanique appliquée, qui se rapproche un peu plus de la réalité, en considérant les corps élastiques, les liens extensibles, les frottements, etc. [...]	A l'économie politique pure fait suite l'économie politique appliquée, qui ne considère plus seulement l'*homo oeconomicus*, mais d'autres êtres se rapprochant plus de l'homme réel. [...]

From this passage it appears that, according to Pareto, for each of the relevant concepts, theories, principles, and subdivisions of the science of economics one can find a corresponding counterpart within the science of mechanics. Thus, the concept of a rational agent in economics (Pareto's "*homo oeconomicus*") corresponds to the concept of a "material point" or "body" in mechanics; similarly, the concept of an economy consisting of a finite number of interacting agents (Pareto's "système économique" or "société") corresponds to the concept of a discrete "mechanical system" comprising a finite number of interrelated material points or bodies. In the same vein, the distinction between "pure" and "applied" economics parallels the distinction between "rational" and "applied" mechanics. But then, more technically and, from our point of view, more interestingly, within the field of pure economics the study of the manifestations of "ophelimity" (since 1896, Pareto's word for what was then, and still is nowadays, currently called "utility") reflects the analogous study of the actions of "forces" in rational mechanics; the subdivision of pure economics into two parts, "statics" and "dynamics", reproduces the analogous subdivision in rational mechanics; the concept of an "equilibrium" in economic statics corresponds to the analogous concept in mechanical statics. Finally, according to Pareto, even d'Alembert's Principle, which is one of the most famous principles of classical mechanics, would possess its own counterpart in the field of pure economics; but, as Pareto hastens to warn us, "our knowledge about this subject is still highly imperfect", so that "at present we are only able to catch a glimpse of a similar principle in economics" (Pareto (1896-97, Vol. II, pp. 9-10)).

The above list of correspondences between the two sciences is much more than a rhetorical device used to buttress the relatively recent discipline of mathematical economics by resorting to the long-established, undisputable prestige of classical mechanics. Indeed, what Pareto is explicitly suggesting is that the asserted analogy between mechanics and economics can be *analytically* exploited in order to pursue a well-defined end, which can be summarized as follows.

According to Pareto, as we have seen, pure economics, like rational mechanics, can be subdivided into two parts: statics and dynamics. But while in mechanics both parts have reached a satisfactory degree of theoretical development, in economics, instead, there exists a fundamental asymmetry between the two subdisciplines: economic statics, which Pareto essentially identifies with Walrasian General Equilibrium Theory and its possible extensions, already rests on a relatively firm theoretical ground; economic dynamics, on the contrary, still is in its infancy: it is a wild land whose systematic exploration has yet to be started. Given such situation, the social scientist who is willing to draw from the mechanical analogy all its *analytical* implications for the development of theoretical economics is forced to adopt a diversified strategy.

As far as statics is concerned, in Pareto's opinion, the starting point is to honestly acknowledge that the fundamental equations of general equilibrium (in a pure exchange economy), which ought to be labelled "Walras' equations" in honor of their discoverer, are formally identical to the equations which can be obtained by applying the well-known Principle of Virtual Works (also known as the Principle of Virtual Displacements, or Powers, or Velocities) to a suitably constructed problem in mechanical statics (see, e.g., Pareto (1892-93, Part I, p. 415; Part II, p. 497), (1896-7, Vol. I, pp. 24-25), (1902, p. 151)). But then the course to be followed in the further development of economic statics is readily traced: as in mechanics, after the publication of Lagrange's *Mécanique analitique* (1788), the Principle of Virtual Works has become the foundation of the whole of mechanical statics; so in economics, after the publication of Walras' *Eléments d'économie politique pure* (1874-77), "Walras' equations", which can be regarded as the economic analogue of Lagrange's equations in mechanics, are to become the foundation of the whole of economic statics. Moreover, due to the structural similarities between the two subdisciplines at the foundational level, in the process of elaboration of economic statics it will be possible to bodily import from mechanical statics some theorems which have already been proved to hold in the latter context.

As far as dynamics is concerned, however, the stance taken by Pareto is necessarily much subtler than the one taken in the case of statics: for he is aware that the two contexts are so different that it would be unwarranted to try to apply the same strategy in either case. In Pareto's opinion, the main difficulty in this respect arises from the following fact: while economic dynamics is admittedly underdeveloped and almost inexistent, mechanical dynamics is instead a powerful theoretical system, based on solid foundations (specifically, on a set of well-established postulates and laws, such as the so-called Fundamental Law of Dynamics, often identified with Newton's Second Law). But then, owing to the poor state of economic dynamics, there is evidently no basic postulate or fundamental law already available in this field to which the social scientist can think to ascribe a role similar to the one that the natural scientist can legitimately attribute to, say, the Fundamental Law of Dynamics in the field of mechanics; this means, however, that in the dynamical case, unlike in the statical one, the analogy between mechanics and economics does not seem to hold, at least at first sight, at the foundational level.

Yet Pareto is apparently able to dodge this difficulty by resorting to an ingenious device, based on the well-known principle of classical mechanics known as d'Alembert's Principle. According to a popular interpretation, wholeheartedly embraced by Pareto, d'Alembert's Principle would play a relevant role, both heuristic and unifying, in rational mechanics, for - to use Pareto's words - it would "allow one to reduce all dynamical question to a question of statics". Suppose now that, in the field of economics, one could discover a principle analogous to d'Alembert's Principle in mechanics. Then, account being taken of the satisfactory stage of elaboration already reached by economic statics, one might reasonably hope to develop the still weak or inexistent theory of economic dynamics by simply applying the newly discovered economic analogue of d'Alembert's Principle to the already available theory of economic statics. This is precisely the roundabout route suggested by Pareto in order to avoid the obstacles hampering the more direct use of the mechanical analogy in the context of economic dynamics; and this is the reason why he will spend so much time over the two decades he will devote to the development of theoretical economics in order to find the missing economic principle that might take the place of d'Alembert's Principle in mechanics.

At first sight, the methodological position taken by Pareto on mechanics and economics may seem to lend some support to a controversial thesis about neoclassical economics put forward by a number of historians and methodologists of economics, from among whom Mirowski stands out at least for the vehemence of his argumentation. According to Mirowski (1989, p. 9),

> [t]he Marginalists appropriated the mathematical formalisms of mid-nineteenth-century energy physics, [...] made them their own by changing the labels on the variables, and then trumpeted the triumph of a truly 'scientific economics'. Utility became the analogue of potential energy; the budget constraint became the slightly altered analogue of kinetic energy; and the Marginalist Revolutionaries marched off to do battle with classical, Historicist, and Marxian economists.

But the idea that neoclassical economics is the outcome of a slavish imitation of "mid-nineteenth-century energy physics", which is almost coextensive with mid-nineteenth-century mechanics, is completely wrong, even if, somewhat paradoxically, such idea is superficially justified by many rash statements made by the very same founders of, and early contributors to, neoclassical economics. The fact is that, in the anxiety to make their position stronger towards their fellow economists, as well as more acceptable to the scientific community at large, early neoclassical economists, especially those leaning towards the newly born discipline of mathematical economics, were ready both to emphasize all the existing formal similarities and, what is worse, to forge inexisting substantive similarities between economics, on the one hand, and the much sounder and more reputable physical sciences, on the other.

2 Statics, dynamics, and equilibrium in classical rational mechanics

In this Section only those definitions, assumptions, and propositions pertaining to classical rational mechanics will be considered which are strictly necessary to the understanding of Pareto's arguments and the related discussion in the sequel.

A *material point* P is characterized by its position in space and its mass. Given an observer located in a fixed origin O in ordinary 3-space, let $P - O$ (or, shortly, P) denote the *position* of point P with respect to the given spatial frame of reference, which in classical mechanics is taken to be absolute. Henceforth the symbol P will be indifferently used to denote a point as well as its position in space. P is a vector quantity in 3-space. Let m denote the *mass* of point P. In classical mechanics m is a scalar quantity whose measure is taken to be an invariable characteristic of P. If P' and P'' are two distinct positions taken by point P, their vector difference $P'' - P'$ is called the *displacement* of point P from the *initial position* P' to the *final position* P''. An *infinitesimal displacement* of point P is denoted dP.

Let $t \in \mathbf{R}$ be a time parameter. In classical mechanics the time frame of reference is taken to be absolute as well. The position of a material point P can be viewed as a function of time; such a vector-valued function is said to describe the *motion* of P . The image-set of this function is called the *trajectory* of P. The function $P(\cdot)$ is assumed to be twice continuously differentiable with respect to time.

The vector quantity $\mathbf{v} \equiv \dot{P} \equiv \frac{dP}{dt}$ is called the *velocity* of P at a given instant. The scalar quantity $v \equiv (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} \equiv \|\mathbf{v}\|$ is called the *speed* of P. (The symbol $\cdot$ denotes the scalar product of vectors; the symbol $\|\|$ denotes the modulus of the vector included between the double vertical bars.) The vector quantity $\mathbf{a} \equiv \ddot{P} \equiv \frac{d^2P}{dt^2}$ is called the *acceleration* of P at a given instant.

A *discrete material system* is a finite collection of material points P_i (i = *1, 2, ..., N*). Only discrete systems will be considered in the following, for these are the only mechanical systems that are explicitly taken into consideration by Pareto for the purposes of comparison with economic systems (similarly assumed to consist of a finite number of agents).

The displacements and motions of the points belonging to a certain material system may be *free* or *constrained.* A constraint may *internal* (if it only depends on the mutual relationships among the points of the system) or *external* (otherwise); *smooth* (if it only depends on the geometrical properties of the relations expressing the constraint) or *rough* (otherwise); *one-sided* (e.g., when a point is constrained to stay in the region lying on one side of a surface, in which case the relations expressing the constraint take the form of inequalities) or *two-sided* (e.g., when a point is constrained to stay on a surface, in which case the relations expressing the constraint take the form of equations); *fixed* (if it does not depend on time) or *mobile* (otherwise).

A *virtual displacement* of point P_i, denoted δP_i, is an infinitesimal displacement of P_i, conforming to the constraints to which P_i is subject at a given instant (the constraints being taken as fixed, even if they are mobile, for the purposes of differentiation). A virtual displacement δP_i is said to be *reversible* if $-\delta P_i$ is a virtual displacement as well, *irreversible* otherwise. A virtual displacement of point P_i is necessarily reversible if P_i is subject to two-sided constraints. The vector quantity $\mathbf{v}_i^{'} \equiv \frac{\delta P_i}{\delta t}$ is the *virtual velocity* of point P_i at a given instant; such virtual velocity is said to be *reversible* or *irreversible* according to whether the corresponding virtual displacement is *reversible* or *irreversible.*

Let $\mathbf{F}_i$ denote the *force* or, more generally, the *resultant* (i.e., the vector sum) *of the forces* acting on the point P_i. $\mathbf{F}_i$ is a vector quantity in $\mathbf{R}^3$. $\mathbf{F}_i$ may be thought of as the sum of an *active force*, $\mathbf{F}_i^{(a)}$, and a *reactive force* due to the constraints to which P_i is subject, $\mathbf{F}_i^{(c)}$, that is: $\mathbf{F}_i \equiv \mathbf{F}_i^{(a)} + \mathbf{F}_i^{(c)}$. Of course $\mathbf{F}_i^{(c)} = 0$ if the point P_i is free.

In general, a force $\mathbf{F}_i$ acting on the point P_i may be thought of as depending on the position of the point, P_i, its velocity, $\mathbf{v}_i$, and possibly other factors related to the physical characteristics of both the point itself and the other points or bodies interacting with it; such factors may in turn depend on time. Hence, by collectively grouping all the factors different from position and velocity under the time parameter, we can functionally represent a force as: $\mathbf{F}_i = \mathbf{F}_i(P_i, \mathbf{v}_i, t)$. Yet, in some important cases, a force acting on a point may only depend on its position, being unaffected by the velocity of the point and by time. In such a case, the force is called *positional.* A positional force may be regarded as a vector field from a suitable domain in 3-space to 3-space; with a slight abuse of notation, such a vector field will be functionally represented as: $\mathbf{F}_i = \mathbf{F}_i(P_i)$. A positional force is called *conservative* if there exists a scalar field, defined on a non-empty set in 3-space, such that, for each position in that set, the vector field representing the force coincides with the gradient of the scalar field, which is then called the *potential* of the force; in symbols: $\mathbf{F}_i(P_i) = \nabla U_i(P_i)$, where U_i is the potential.

The differential expression $d^*W_i \equiv \mathbf{F}_i \cdot dP_i$ represents the *elementary work* done by the force $\mathbf{F}_i$ acting on the point P_i in bringing about the infinitesimal displacement dP_i (the asterisk after the symbol of differentiation is meant to signal that d^*W_i need not be an exact differential). If, in the above expression, the infinitesimal displacement dP_i is replaced by the virtual displacement δP_i, one obtains the differential expression $\delta^*W_i \equiv \mathbf{F}_i \cdot \delta P_i$, representing the *virtual work* done by the force $\mathbf{F}_i$ acting on the point P_i. The expression $\Pi_i \equiv \frac{d^*W_i}{dt} \equiv \mathbf{F}_i \cdot \frac{dP_i}{dt} \equiv \mathbf{F}_i \cdot \mathbf{v}_i$ represents the *instantaneous power* of the force $\mathbf{F}_i$ acting on the point P_i at a given instant. If, in this expression, the velocity $\frac{dP_i}{dt} \equiv \mathbf{v}_i$ is replaced by the virtual velocity $\frac{\delta P_i}{\delta t} \equiv \mathbf{v}_i'$, one obtains the expression $\Pi_i' \equiv \frac{\delta^*W_i}{\delta t} \equiv \mathbf{F}_i \cdot \frac{\delta P_i}{\delta t} \equiv \mathbf{F}_i \cdot \mathbf{v}_i'$, representing the *virtual power* of the force $\mathbf{F}_i$ acting on the point P_i at a given instant.

We are now in a position to introduce the Principle of Virtual Works (or Virtual Displacements), which is perhaps the most fundamental result in mechanical statics: Given any discrete material system consisting of a finite number N of points P_i, subject to smooth, but otherwise arbitrary, constraints, let $\mathbf{F}_i^{(a)}$ be the active force acting on P_i and $\delta^*W_i^{(a)} \equiv \mathbf{F}_i^{(a)} \cdot \delta P_i$ the virtual work done by the active force $\mathbf{F}_i^{(a)}$ $(i = 1,\ 2,\ \ldots,\ N)$. Then the following differential relation holds: $\delta^*W^{(a)} \equiv \sum_{i=1}^{N} \delta^*W_i^{(a)} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(a)} \cdot \delta P_i \leq 0$. Moreover, if all the displacements are reversible (what is implied by the constraints being two-sided), then the above relation holds as an equation.

It is convenient to give separate expression to the two cases jointly considered under the Principle of Virtual Works. In fact, according to whether only reversible or also irreversible virtual displacements are allowed to occur, one can derive from the above Principle two separate results: on the one hand, the SYMBOLIC EQUATION OF STATICS

$$\delta^* W^{(a)} \equiv \sum_{i=1}^{N} \delta^* W_i^{(a)} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(a)} \cdot \delta P_i = 0 \quad , \tag{1}$$

holding for all discrete systems of material points subject to two-sided smooth constraints (hence for which only reversible virtual displacements are admissible); on the other, the SYMBOLIC RELATION OF STATICS

$$\delta^* W^{(a)} \equiv \sum_{i=1}^{N} \delta^* W_i^{(a)} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(a)} \cdot \delta P_i \leq 0 \quad , \tag{2}$$

holding for all discrete systems of material points subject to arbitrary (possibly one-sided) smooth constraints (hence for which irreversible virtual displacements are possible as well).

By replacing the virtual displacements δP_i with the corresponding virtual velocities $\frac{\delta P_i}{\delta t} \equiv \mathbf{v}_i^{'}$, hence the virtual works of the active forces $\delta^* W_i^{(a)} \equiv \mathbf{F}_i^{(a)} \cdot \delta P_i$ with the corresponding virtual powers $\Pi_i^{(a)'} \equiv \frac{\delta^* W_i^{(a)}}{\delta t} \equiv \mathbf{F}_i^{(a)} \cdot \frac{\delta P_i}{\delta t} \equiv \mathbf{F}_i^{(a)} \cdot \mathbf{v}_i^{'}$, in the symbolic equation and relation of statics, one obtains the following pair of relations

$$\Pi^{(a)'} \equiv \sum_{i=1}^{N} \Pi_i^{(a)'} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(a)} \cdot \mathbf{v}_i^{'} = 0 \qquad \text{(for reversible virtual velocities)} \tag{3}$$

and

$$\Pi^{(a)'} \equiv \sum_{i=1}^{N} \Pi_i^{(a)'} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(a)} \cdot \mathbf{v}_i^{'} \leq 0 \qquad \text{(for arbitrary virtual velocities),} \tag{4}$$

which jointly provide an alternative formulation of the Principle of Virtual Works, often referred to as the Principle of Virtual Powers (or Velocities). The two versions of the Principle are fully equivalent. Yet, as long as the Principle is regarded as a fundamental result of mechanical *statics* and is used for the purposes of *equilibrium* *analysis*, it seems preferable to hold to the first version, based on virtual displacements and works, rather than to the second, based on virtual velocities and powers. In fact, for methodological reasons, statics and equilibrium analysis ought to be kept as free as possible from concepts depending on time; but while virtual displacements and works do meet this condition, virtual velocities and powers don't. As we shall see, however, the second formulation of the Principle proves useful in discussing the relationships between statics and dynamics.

The Principle of Virtual Works or, what is the same, the Symbolic Equation and Relation of Statics express the equilibrium conditions of a discrete material system. For the equilibrium of a system it is necessary that the resultant of the forces acting on every point P_i of the system be nil, that is

$$\mathbf{F}_i \equiv \mathbf{F}_i^{(a)} + \mathbf{F}_i^{(c)} = 0 \quad \text{or} \quad \mathbf{F}_i^{(a)} = -\mathbf{F}_i^{(c)} \ . \tag{5}$$

In view of this, the relations (1) and (2) can be rewritten as:

$$\delta^* W^{(c)} \equiv \sum_{i=1}^{N} \delta^* W_i^{(c)} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(c)} \cdot \delta P_i = 0 \quad \text{(for reversible virtual displacements)} \tag{6}$$

and

$$\delta^* W^{(c)} \equiv \sum_{i=1}^{N} \delta^* W_i^{(c)} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(c)} \cdot \delta P_i \geq 0 \quad \text{(for arbitrary virtual displacements),} \tag{7}$$

where $\delta^* W_i^{(c)}$ is the virtual work done by the reactive force $\mathbf{F}_i^{(c)}$ due to the constraints to which P_i is subject.

Similarly, the relations (3) and (4) can be rewritten as:

$$\Pi^{(c)\prime} \equiv \sum_{i=1}^{N} \Pi_i^{(c)\prime} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(c)} \cdot \mathbf{v}_i' = 0 \qquad \text{(for reversible virtual velocities)} \tag{8}$$

and

$$\Pi^{(c)\prime} \equiv \sum_{i=1}^{N} \Pi_i^{(c)\prime} \equiv \sum_{i=1}^{N} \mathbf{F}_i^{(c)} \cdot \mathbf{v}_i' \geq 0 \qquad \text{(for arbitrary virtual velocities),} \tag{9}$$

where $\Pi^{(c)\prime}$ is the virtual power of the reactive force $\mathbf{F}_i^{(c)}$ due to the constraints to which P_i is subject.

Suppose now, in particular, that the material system consists of points subject to two-sided, smooth constraints, so that the Principle of Virtual Works is rendered by the Symbolic Equation of Statics and the constraints are expressed by equations rather than inequalities. Then, once a set of coordinates suitable for defining the position of the system has been chosen, granting that at least an equilibrium exists, equation (1) and the equations expressing the constraints can be jointly used to determine the equilibrium position(s) of the system in terms of the chosen coordinates. At the end of this Section we shall illustrate how the equilibrium position(s) of a system can be determined by means of a very simple example, that is, by deriving the equilibrium conditions for a constrained system consisting of one single point. In spite of its simplicity, this example is instructive for, as will be seen, it is precisely this oversimplified situation that Pareto had in mind when trying to justify the alleged similarities, or even identities, between mechanics and economics concerning both static (equilibrium) and dynamic analysis.

Let us now turn to mechanical dynamics. In this context the central theoretical issue is to explain the motion of a material point or system; for this reason we shall now speak of *mobile* material points and systems.

For any mobile material point P_i, free or constrained, the FUNDAMENTAL LAW OF DYNAMICS is expressed by the following vector equation:

$$\mathbf{F}_i \equiv \mathbf{F}_i^{(a)} + \mathbf{F}_i^{(c)} = m_i \mathbf{a}_i \quad , \tag{10}$$

asserting that the acceleration $\mathbf{a}_i$ of the mobile point P_i at any given instant is directed as, and proportional to, the resultant $\mathbf{F}_i$ of the forces (both active, $\mathbf{F}_i^{(a)}$, and reactive, $\mathbf{F}_i^{(c)}$) acting on P_i at that instant, the factor of proportionality being represented by the reciprocal of the mass m_i of P_i. In this context the force $\mathbf{F}_i$ acting on P_i may be qualified as the *driving force* of the mobile point P_i. As can be seen, the Fundamental Law of Dynamics implies Newton's Second Law, with which it is frequently identified.

By simple manipulation, equation (10) can be rewritten as:

$$\mathbf{F}_i^{(a)} - m_i\mathbf{a}_i = -\mathbf{F}_i^{(c)} \quad , \tag{11}$$

where, in the wake of d'Alembert, $-m_i\mathbf{a}_i$ and $(\mathbf{F}_i^{(a)} - m_i\mathbf{a}_i)$ are said to represent the *force of inertia* and the *lost force* acting on P_i, respectively. (The second expression can be explained as follows. From the identity $\mathbf{F}_i^{(a)} \equiv m_i\mathbf{a}_i + (\mathbf{F}_i^{(a)} - m_i\mathbf{a}_i)$ it follows that any active force $\mathbf{F}_i^{(a)}$ can be viewed as the sum of two parts: while the first one, $m_i\mathbf{a}_i$, is the driving force which would cause the acceleration $\mathbf{a}_i$ if the material point P_i on which it acts were free, the second one, $\mathbf{F}_i^{(a)} - m_i\mathbf{a}_i$, is instead "lost" for the purposes of motion.) By employing this terminology, the above equation can be interpreted as asserting that, for each material point P_i, the lost force acting on P_i is the opposite of the reactive force due to the constraints to which P_i is subject.

Now, in order to develop mechanical dynamics, the Fundamental Law needs only to be supplemented by the following weak postulate, which is almost self-evident: in any discrete mobile system consisting of material points subject to smooth constraints, the forces of reaction due to the constraints keep satisfying the equilibrium conditions, as expressed by either the equation (8) or the relation (9), as the case may be, even when the motion of the system is explicitly allowed for and taken into account. Then, by substituting equation (11) into relations (8) and (9), one obtains the following pair of relations:

$$\sum_{i=1}^{N}(\mathbf{F}_i^{(a)} - m_i\mathbf{a}_i)\cdot\mathbf{v}_i^{'} = 0 \quad \text{(for reversible virtual velocities)} \tag{12}$$

and

$$\sum_{i=1}^{N}(\mathbf{F}_i^{(a)} - m_i\mathbf{a}_i)\cdot\mathbf{v}_i^{'} \leq 0 \quad \text{(for arbitrary virtual velocities),} \tag{13}$$

which hold for all discrete mobile systems consisting of material points subject to smooth constraints, under the specified conditions, and which are commonly known as the Symbolic Equation and the Symbolic Relation of Dynamics, respectively.

Suppose now, in particular, that the material system consists of points subject to two-sided, smooth constraints, so that the Symbolic Equation of Dynamics holds and the constraints are expressed by equations rather than inequalities. Then, once a set of coordinates suitable for defining the position of the system has been chosen, assuming the functions entering equation (12) to be sufficiently regular, equation (12) and the equations expressing the constraints can be jointly used to determine the motion of the system in terms of the chosen coordinates: specifically, this means to determine the values of the coordinates as functions of time and of constants which, in view of the fact that equation (12) is a differential equation of the second order, must be determined by assigning a suitable set of initial conditions. At the end of this Section we shall illustrate the dynamic problem of the calculation of the motion of a mobile system by resorting to the same example (namely, a simple one-point system) as the one that will be used to discuss the static equilibrium problem. But before turning to this specific example, it is necessary to examine a few remaining issues of a general character, concerning the relationships between statics and dynamics in mechanics.

If we set $\mathbf{a}_i = 0$ ($i = 1, 2, \ldots, N$) in the Symbolic Equation and Relation of Dynamics, that is, in relations (12) and (13), we immediately get the relations (3) and (4) which, as will be recalled, jointly provide that alternative formulation of the statical Principle of Virtual Works that is generally known as the Principle of Virtual Powers (or Velocities); moreover, if we replace $\mathbf{v}_i'$ by δP_i ($i = 1, 2, \ldots, N$) in the latter relations, we get relations (1) and (2), that is, the Symbolic Equation and Relation of Statics, respectively. This means that the fundamental relations governing the static equilibrium of a material system can be viewed as a special case of the fundamental relations governing its dynamic motion, namely, as that special case that corresponds to a nil acceleration, hence a constant velocity, for each point belonging to the system; but a constant velocity, as is well-known (as well as implied by Newton's First Law), is precisely the characteristic property of the only sort of motion that is consistent with static equilibrium.

It has just been shown that the Symbolic Equation and Relation of Statics can be derived from the Symbolic Equation and Relation of Dynamics; but it can also be shown that, in a special sense, the reverse derivation is possible as well. Since d'Alembert's Principle is instrumental in the latter derivation, we shall start by explaining the meaning of this Principle. To this end, let us go back once again to the Fundamental Law of Dynamics, specifically to the version of that Law contained in equation (11), which we reproduce here for convenience:

$$\mathbf{F}_i^{(a)} - m_i \mathbf{a}_i = -\mathbf{F}_i^{(c)} \quad .$$

Recalling now equation (5), according to which a necessary condition for an equilibrium of a discrete constrained material system is that the active force acting on each point P_i belonging to the system be the opposite of the reactive force due to the constraints to which that point is subject, we can suggest the following interpretation of equation (11): during the motion of a mobile system, for each material point P_i belonging to the system, the lost force, $\mathbf{F}_i^{(a)} - m_i \mathbf{a}_i$, exactly offsets the reactive force due to the constraints to which that point is subject, $\mathbf{F}_i^{(c)}$, thereby satisfying, instant by instant, the equilibrium conditions. From this interpretation it follows that the necessary condition for an equilibrium of a material system can still be regarded as being satisfied, instant by instant, during the motion of the system, provided that the active forces appearing in the original statical formulation of the equilibrium condition be replaced by the lost forces in the new dynamical version.

The above discussion naturally leads to the following operational principle, which is generally referred to as D'ALEMBERT'S PRINCIPLE: One can pass from the static equations (relations) determining the equilibrium position(s) of a material system to the dynamic equations (relations) governing its motion simply by substituting the lost forces, $\mathbf{F}_i^{(a)} - m_i \mathbf{a}_i$, for the active forces, $\mathbf{F}_i^{(a)}$, appearing in the former equations (relations).

From a heuristic point of view, d'Alembert's Principle is very useful for, by providing a very simple, automatic rule for obtaining the equations of motion of a system from its equilibrium equations, it allows one to reduce any dynamical problem to the corresponding statical problem. To illustrate this aspect, it is probably enough to mention that, by applying d'Alembert's Principle, relations (12) and (13) can be derived from relations (3) and (4), so that, in the last analysis, the Symbolic Equation and Relation of Dynamics can be traced back to the Symbolic Equation and Relation of Statics.Yet, if the pragmatic significance of d'Alembert's Principle should not be underestimated, at the same time its theoretical relevance should not be overrated: in fact, from a logical and epistemological point of view, this Principle adds nothing to the Symbolic Equation (or Relation) of Dynamics, since both depend on the very same premises (namely, the Fundamental Law of Dynamics and the postulate mentioned above); it is on these premises, therefore, that the whole burden of the explanation rests. This remark will prove relevant later on, when discussing Pareto's position on d'Alembert's Principle.

Before concluding this Section, let us illustrate the statical problem of the determination of the equilibrium position(s) of a material system as well as the dynamical problem of the calculation of its motion by means of a very simple example. Precisely, let us consider a material system consisting of one single point P with mass m, which is constrained to move on a smooth surface in 3-space and is affected by an active force $\mathbf{F}^{(a)}$, assumed to be positional; hence, there are no internal constraints, the only external constraint is two-sided, and the active force only depends on the position, i.e., $\mathbf{F}^{(a)} = \mathbf{F}^{(a)}(P)$.

Given a Cartesian frame of reference $(O\mathbf{ijk})$, where O is the origin, while $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors directed along the three orthogonal axes, let the position of point P be given by the three Cartesian coordinates x, y, z, that is, $P(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Further, while in general the surface on which P is constrained to move may itself be mobile, in discussing the statical problem we are compelled to take it as fixed (at a certain instant, which need not be specified). Hence, in this context, we shall assume the constraint to be given by the scalar equation

$$f(x, y, z) = 0 \ , \tag{14}$$

where f is a continuosly differentiable scalar field having the three coordinates as its only arguments. In this case the Symbolic Equation of Statics, which renders the Principle of Virtual Works, is simply given by

$$\mathbf{F}^{(a)} \cdot \delta P = 0 \ .$$

But any virtual displacement δP is necessarily tangent to the surface at P, hence orthogonal to the vector normal to the surface at P, which, in its turn, is nothing other than the gradient of f, ∇f, evaluated at P. Hence, at any equilibrium position of the one-point system, $\mathbf{F}^{(a)}$ must be parallel to ∇f, which means that the vector equation

$$\mathbf{F}^{(a)} = \lambda \nabla f \tag{15}$$

or, equivalently, the three scalar equations

$$\begin{aligned} F_x^{(a)} &= \lambda \frac{\partial f}{\partial x} \\ F_y^{(a)} &= \lambda \frac{\partial f}{\partial y} \\ F_z^{(a)} &= \lambda \frac{\partial f}{\partial z} \end{aligned}$$

must hold, where λ is a scalar multiplier and $F_x^{(a)}$, $F_y^{(a)}$, and $F_z^{(a)}$ are the Cartesian components of the active force along the three axes.

Granting sufficient regularity conditions, equations (14) and (15) jointly allow one to determine the equilibrium position(s) of the system, in terms of the equilibrium values of the coordinates, as well as the associated equilibrium value(s) of the multiplier. Let (P^*, λ^*) be any such equilibrium pair, where $P^* = P(x^*, y^*, z^*)$. Then the equilibrium value of the active force is $\mathbf{F}^{(a)*} = \lambda^* \nabla f(P^*)$, the equilibrium value of the reaction of the constraint being of course just the opposite, that is, $\mathbf{F}^{(c)*} = -\mathbf{F}^{(a)*} = -\lambda^* \nabla f(P^*)$.

With reference to the same one-point material system, the dynamical problem can be dealt with as follows. The motion of the system, which is here regarded as a mobile system, is given by specifying the three coordinates as functions of time, namely, $x = x(t)$, $y = y(t)$, and $z = z(t)$, so that $P(t) = P(x(t), y(t), z(t))$. The active force, being positional, is still given by the equation $\mathbf{F}^{(a)} = \mathbf{F}^{(a)}(P)$. In this context, however, we must explicitly allow for the mobility of the constraint; hence, we shall assume the constraint to be given by the scalar equation

$$g(x, y, z; t) = 0 \ , \tag{16}$$

where g is a continuously differentiable scalar field depending not only on the three positional coordinates, but also on time. Let $\nabla_{(xyz\ |\ t)}g$ denote the gradient of g with respect to the three positional coordinates, taking time as fixed at t. Of course, with this convention, if $g(x, y, z; t) = f(x, y, z)$, then also $\nabla_{(xyz\ |\ t)}g = \nabla f$.

By virtue of d'Alembert's Principle, the laws of motion of the system can be immediately derived from the static equilibrium equations. To this end, it is sufficient to replace $\mathbf{F}^{(a)}$, ∇f, and λ by $(\mathbf{F}^{(a)} - m\mathbf{a})$, $\nabla_{(xyz\ |\ t)}g$, and μ, respectively, in equation (15), where $\mathbf{a} = \ddot{P} = \frac{d^2P}{dt^2}$ is the acceleration of P, and μ is a scalar multiplier, to get the vector equation

$$\mathbf{F}^{(a)} - m\mathbf{a} = \mu\nabla_{(xyz\ |\ t)}g \tag{17}$$

or, equivalently, the three scalar equations

$$\begin{aligned}
F_x^{(a)} - ma_x &= \mu\frac{\partial g}{\partial x} \\
F_y^{(a)} - ma_y &= \mu\frac{\partial g}{\partial y} \\
F_z^{(a)} - ma_z &= \mu\frac{\partial g}{\partial z}
\end{aligned}$$

where a_x, a_y, and a_z are the Cartesian components of $\mathbf{a}$, and all the partial derivatives of g are evaluated at t.

By jointly considering equations (16) and (17), after eliminating μ, one can determine the three coordinates as functions of time and of six constants, which can in turn be determined by taking the initial conditions into account.Of course, if the constraint is not a constant function of time, $\nabla_{(xyz\ |\ t)}g$ is not independent of t, so that the law of motion of the system cannot be determined unless the dynamical law is known which governs the evolution of the constraint over time. The latter remark will prove relevant later on, when discussing Pareto's position on economic dynamics.

3 Statics and equilibrium in mechanics and economics

In this Section we shall discuss Pareto's contention that "rational mechanics" and "pure economics" exhibit "amazing analogies" in the field of statics, especially of equilibrium analysis. To make his point Pareto resorts to a number of pairwise comparisons, where a concept, assumption, or proposition pertaining to mechanical statics is compared with a concept, assumption, or proposition pertaining to economic statics. As far as the economic side of the comparisons is concerned, the situations discussed by Pareto almost invariably refer to that part of General Equilibrium Theory that deals with competitive pure-exchange economies. This fact deserves some comment.

In his early works, where Walras' influence is stronger, and particularly in the introductory part of the *Cours* (1896-97), a sort of summary of Walrasian General Equilibrium Theory called "Principes d'économie politique pure", Pareto, in the wake of Walras (1874-77), regards the pure-exchange model merely as the first and simplest of a sequence of nested models of increasing complexity, progressively extending their scope to encompass the phenomena of production, capital formation, credit, and money; "free competition", though occupying a central place in the analysis, is regarded as one of a number of alternative institutional settings, including a "socialistic" organization of the economy which is already explicitly discussed in the *Cours*. In his later works Pareto will further strengthen this attitude: over the years the pure-exchange competitive model will increasingly be viewed as an instance of a much more general and abstract model of rational choice and social interaction, which will eventually become the explicit subject of Pareto's investigations (see, e.g., Pareto (1902, pp. 139-140), (1909, p. 207, and App., p. 543, fn. 1)).

Yet, all this being said, Pareto's almost exclusive concentration on the pure-exchange competitive equilibrium model in discussing the relastionships between mechanical and economic statics can be explained as follows. The first reason is simple. In spite of his ever increasing ambitions to abstractness and generality, for the whole of his scientific life Pareto will continue to regard the pure-exchange competitive model as the fundamental prototype on which to build the whole of scientific economics: in fact, it is precisely to this model that Pareto invariably refers when it comes to specifying the economic equivalent of the mechanical equilibrium model ascribed to Lagrange ([1]). The second reason is subtler. As we shall see, Pareto's "proof" of the asserted equivalence between mechanical and economic equilibrium is far from convincing. But, by sticking to the pure-exchange competitive case, he is at least able to identify a few seeming similarities between some of the equations defining the equilibrium conditions in the two sciences. On the contrary, had he used a more general model as his frame of reference on the economic side of the comparison, he probably could not have reached even this limited result[2].

The place where Pareto's stance on this issue is most clearly stated is probably represented by the following passage of the *Cours*:

> Considérons trois biens économiques: A, B, C. Tirons trois axes rectangulaires: x, y, z. Nous porterons sur x, les quantités de A, sur y, les quantités de B, et sur z, les quantités de C. Soit a la quantité de A que possède un individu quand il n'a ni B ni C, b la quantité de B que

[1]See, in particular, Pareto (1896-97, Vol. I, pp. 24-25, (**59**)[1]). See also Pareto (1892-93, Part I, p. 415) and (1902, p. 151).

[2]In effect, from his early works to his latest economic writings, Pareto will unceasingly strive to generalize the pure-exchange competitive equilibrium model, where prices are taken as fixed parameters by the individual traders in deciding the global amounts of the commodities to trade, by developing an allegedly more general non-competitive model, where prices are allowed to vary with the "successive portions" of the commodities that are being traded (for this reason the extended model is often ambiguously referred to by Pareto as the model with "non-constant" or "variable prices"; see, e.g., Pareto (1892-93, Part I, p. 414) and (1909, App., p.566 ff.)). Over the years the model with "variable prices" will reach a more than rudimentary degree of formal elaboration; yet, in spite of this, it will never be used by Pareto as the economic standard of reference in discussing the relationships between mechanics and economics, probably for the reasons suggested in the text. In any case, following Pareto, we shall completely disregard this model in the following discussion, focussing attention almost exclusively on the pure-exchange competitive model.

possède l'individu quand il n'a ni A ni C, enfin, c la quantité de C que possède l'individu quand il n'a ni A ni B. On aura

$$\frac{a}{b} = \frac{p_b}{p_a} \ , \qquad \frac{a}{c} = \frac{p_c}{p_a} \ ;$$

c'est-à-dire qu'on pourra poser

$$a = \frac{h}{p_a} \ , \qquad b = \frac{h}{p_b} \ , \qquad c = \frac{h}{p_c} \ .$$

Le phénomène de l'échange pourra se représenter par le mouvement d'un point matériel, sollicité parallèlement aux axes coordonnés par les forces φ_a, φ_b, φ_c, et qui doit se mouvoir sur un plan déterminé par la condition qu'il coupe les axes coordonnés aux distances a, b, c.

L'equation de ce plan est

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \ ; \tag{1}$$

c'est-à-dire

$$p_a x + p_b y + p_c z = h \ , \quad [P1]$$

équation qui correspond à l'équation (1) (**59**[1]).

Soit T la force qui représente la résistance du plan, nous savons que la normale au plan fait avec les axes coordonnés des angles dont les cosinus sont

$$\frac{p_a}{\sqrt{p_a^2 + p_b^2 + p_c^2}} \ , \quad \frac{p_b}{\sqrt{p_a^2 + p_b^2 + p_c^2}} \ , \quad \frac{p_c}{\sqrt{p_a^2 + p_b^2 + p_c^2}} \ ;$$

les équations de l'équilibre seront donc

$$\varphi_a - \frac{p_a}{\sqrt{p_a^2 + p_b^2 + p_c^2}} T = 0 \, , \quad \left[P2'\right]$$

$$\varphi_b - \frac{p_b}{\sqrt{p_a^2 + p_b^2 + p_c^2}} T = 0 \, , \quad \left[P2''\right]$$

$$\varphi_c - \frac{p_c}{\sqrt{p_a^2 + p_b^2 + p_c^2}} T = 0 \, . \quad \left[P2'''\right]$$

En éliminant T, on trove

$$\frac{1}{p_a}\varphi_a = \frac{1}{p_b}\varphi_b = \frac{1}{p_c}\varphi_c \; ; \qquad [P3]$$

c'est-à-dire les équations (4) ($\mathbf{59}^1$)[3].

As can be seen, Pareto's strategy simply consists in introducing a common formal apparatus, for which he then provides a twofold interpretation, alternatively in terms of economic and mechanical concepts. As far as statics is concerned, the "proof" of the substantive similarity, and formal identity, between economics and mechanics would then be provided by the fact that the equilibrium conditions in either science can be expressed by means of the same set of equations, when suitably interpreted.

[3]This passage is contained in an "addition" which, while ideally appended to Sec. **144**, fn. 1, of Vol. I of the *Cours*, is actually printed at the end of Vol. II (Pareto (1896-97, Vol. II, p. 411-412)). The notation used by Pareto in this passage is not entirely consistent with that employed in other passages of both the *Cours* and other writings that will be quoted in the sequel. Such inconsistency is partly due to Pareto's attempt (particularly evident in the present passage) to employ a notation reminiscent of that characteristic of mechanics, which however is not always the most appropriate to deal with economic issues. (In this respect, it is enough to recall the problems related to the intrinsically different dimensionalities of the basic reference spaces in the two sciences: in fact, while in mechanics ordinary space is naturally restricted to at most three dimensions, no such restriction obviously applies to the commodity space, which is the standard reference space in General Equilibrium Theory.) In any case, except for one single instance, which will be pointed out in due course, we shall not try to make Pareto's notation more consistent by changing his symbols. However, we shall strive to bring his presentation closer to current usage by introducing a few new symbols to supplement his original notation and, especially, by freely using vector notation, which Pareto never employed (though he probably had some idea of its potentialities, as can be inferred, e.g., from his reference to the "theory of quaternions" a few lines before the beginning of the passage quoted in the text). When necessary, the equations and relations used by Pareto will be identified by means of the label P followed by a progressive integer between square brackets. Finally, in the passage quoted in the text there is a cross-reference to two equations contained in footnote $(\mathbf{59})^1$; this is the place where the equilibrium conditions for a pure-exchange competitive economy are first introduced and extensively discussed in the *Cours*.

The mathematical objects underlying both interpretations can be further specified as follows. Let the common universe of discourse be represented by the non-negative orthant of Euclidean 3-space, $\mathbf{R}^3_+$. Letting $\mathbf{x} \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{p} \equiv p_a\mathbf{i} + p_b\mathbf{j} + p_c\mathbf{k}$, we shall assume $\mathbf{x} \in \mathbf{R}^3_+$ and $\mathbf{p} \in \mathbf{R}^3_{++}$[4]. Given the scalar $h > 0$, let $B(\mathbf{p}, h) = \left\{\mathbf{x} \in \mathbf{R}^3_+ \mid \mathbf{p} \cdot \mathbf{x} = h\right\}$ be the relevant region of the plane identified by the scalar h and the normal vector $\mathbf{p} \equiv T\mathbf{n}$, where $\mathbf{n}$ is the unit normal to the plane and $T \equiv \|\mathbf{p}\|$. Finally, let $\boldsymbol{\varphi} \equiv \varphi_a\mathbf{i} + \varphi_b\mathbf{j} + \varphi_c\mathbf{k}$ be a vector field from $\mathbf{R}^3_+$ to $\mathbf{R}^3_+$.

Now, the mechanical interpretation of the above formal model is wholly standard. In fact, in this context $\mathbf{x}$ represents the position of a material point, affected by a positional active force $\boldsymbol{\varphi}$ and constrained to move on the planar region $B(\mathbf{p}, h)$; further, letting $\Im(\mathbf{x}) \equiv \mathbf{p} \cdot \mathbf{x} - h$, one has $\nabla\Im(\mathbf{x}) = \mathbf{p} = \textit{constant}$. Hence, Pareto's equilibrium equations $[P1]$ and $[P2]$ can be seen to represent a special case of the static equilibrium conditions for a one-point material system constrained to move on a given surface, conditions that have been discussed at the end of Section 2 (see equations (14) and (15)). Now, let $\mathbf{x}^*$ denote any value of $\mathbf{x}$ satisfying $[P1]$ and $[P2]$; then $\mathbf{x}^*$ can be interpreted as a static equilibrium position of the one-point constrained material system under discussion.

Except for a particular aspect, to which we shall come back presently, also the economic interpretation of the formal model would be standard nowadays (even if it was certainly not so in Pareto's times). In fact, in this context $\mathbf{R}^3_+$ should be taken to represent the consumption set of a certain competitive consumer; further, $\mathbf{x}$ should be interpreted as a consumption bundle, $\mathbf{p}$ as the given price system, h as the given income (or wealth) of the consumer, so that $B(\mathbf{p}, h)$ becomes the consumer's budget set. The preferences of the consumer are represented by the vector field $\boldsymbol{\varphi}$, whose generic component function, φ_k, is interpreted as the elementary ophelimity

[4] Later on in the paper, following Pareto in his changeable use of notation, we shall occasionally redefine $\mathbf{x}$ as $\mathbf{x} \equiv x_a\mathbf{i} + x_b\mathbf{j} + x_c\mathbf{k}$. Sometimes, instead of using a, b, c as the indices of the three coordinate axes, we shall employ the more common notation 1, 2, 3. The numeric notation will be invariably used when the reasoning is extended to $\mathbf{R}^n$, n being an arbitrary integer greater than 3. In this case we shall write $\mathbf{x} \equiv \sum_{k=1}^{n} x_k\mathbf{e}_k$, x_k and $\mathbf{e}_k$ being the k-th Cartesian component of $\mathbf{x}$ and the k-th Cartesian unit vector, respectively, and similarly for the other variables.

(marginal utility[5]) function of commodity k for the consumer concerned ($k = a, b, c$). This, indeed, is the only feature in Pareto's treatment that is unusual with respect to current habits, insomuch as Pareto takes as primitives the elementary ophelimity functions of the various commodities, without necessarily assuming the existence of a total ophelimity function. But, if we provisionally set aside the question of preference representation, we can conclude that the economic problem outlined in the above-quoted passage is nothing but the wholly conventional, and nowadays trivial, problem of the choice of an optimal consumption bundle by a competitive consumer subject to a given budget constraint. And, in effect, assuming sufficient regularity and concavity conditions, the so-called "equilibrium equations" [$P1$] and [$P3$] do identify the standard conditions for an internal optimum in the consumer's choice problem[6]. Let $\mathbf{x}^*$ denote any such optimal consumption choice. Then, letting $\mathbf{x}(\mathbf{p},h)$ denote the consumer's demand correspondence, we have $\mathbf{x}^* \in \mathbf{x}(\mathbf{p},h)$, or simply $\mathbf{x}^* = \mathbf{x}(\mathbf{p},h)$, if the demand correspondence actually turns out to be a function. Finally, for later reference, let $\mathbf{q} \in \mathbf{R}^3_{++}$ represent the consumer's initial endowment of commodities, assumed to be given; then $\mathbf{r}(\mathbf{p},h) \equiv \mathbf{x}(\mathbf{p},h) - \mathbf{q}$ is the consumer's excess demand correspondence (or function)[7]. Since, for given prices $\mathbf{p}$, we have $h = \mathbf{p} \cdot \mathbf{q}$, with a slight abuse of notation we shall also write $\mathbf{x}(\mathbf{p}, \mathbf{q})$ and $\mathbf{r}(\mathbf{p}, \mathbf{q})$ for the consumer's

[5]From 1896 onwards, Pareto will employ the newly-coined expression "ophelimity" to denote a concept similar to what is currently called "utility". Before 1896 the latter expression had been used by Pareto himself in the usual sense; but after that date the term "utility", being replaced by "ophelimity" for the usual purposes, will take on a different meaning. Since in the following we shall refer to works written by Pareto both before and after 1896, we shall be compelled to employ both the word "utility" and the word "ophelimity", as the case may be; it should be clear, however, that "utility" is here employed in its usual sense (that is, in Pareto's original sense).

[6]It should be noted that, while the standard conditions for the equilibrium of a one-point material system are represented by equations [$P1$] and [$P2$], the standard conditions for an optimal consumption choice are instead represented by equations [$P1$] and [$P3$]. From the last sentence of the above-quoted passage it would appear that, in Pareto's view, equations [$P2$] and [$P3$] are interchangeable for the purposes of the present discussion, so that equations [$P1$] and [$P2$] can be taken to identify the so-called "equilibrium conditions" not only in the mechanical, but also in the economic context. As a matter of fact, this idea is not really well-founded, as we shall explain at the end of this Section. Up to that point, however, in order to simplify the exposition, we shall proceed as if equations [$P1$] and [$P2$], rather than [$P1$] and [$P3$], could be legitimately taken to represent the so-called "equilibrium conditions" in the economic case as well.

[7]While vector notation is ours, the symbols x_k, r_k, and q_k are those usually employed by Pareto to denote the consumer's demand for, excess demand for, and endowment of the k-th commodity, respectively.

demand and excess demand correspondence (or function), respectively.

Coming back now to the preference representation issue, it should be noted that, from 1892 onwards, Pareto consistently goes on suggesting that the economic theorist ought to take as his starting point the marginal utility functions for the various commodities, rather than the consumer's total utility function, since, in general, the latter function cannot even be assumed to exist. A sort of psychological justification for this position is offered by Pareto as early as in 1892: in fact, according to Pareto (1892-93, Part I , pp. 414-415), rational consumers, while capable of evaluating the marginal changes in utility induced by marginal changes in the quantity consumed of each individual commodity, are instead generally unable to attach any definite meaning to the total utility associated to the total consumption of a certain commodity or bundle of commodities. In our opinion, however, the real reason why Pareto comes to advocate such position is entirely different from that suggested *ex post facto* by the author himself. In this respect, the first point to stress is that Pareto's idea that a total utility function need not exist, while the marginal utility functions for the various commodities can safely be assumed to exist, has nothing to do with the alternative between "cardinalism" and "ordinalism" in utility theory, even if Pareto himself will do his part to confuse the issues in some of his writings of the first decade of the twentieth century (particularly Pareto (1906) and (1909, App., pp. 539-557)). But to dispel any doubt on this point it would be enough to consider the timing of the events: while the idea of the possible inexistence of a total utility function can be traced back to 1892, Pareto's "ordinalist" stance, as is well known[8], will only mature a few years later, in 1898, to eventually find its first expression in a journal article (actually the draft of a never completed treatise) only in Pareto (1900).

But then, once the field is cleared of all pseudo-motivations, the true explanation of Pareto's position about marginal and total utility can be easily seen to lie in the mechanical dream haunting this author: indeed, Pareto is led to take the peculiar position illustrated above by his desire to show that, in making his choices, a rational consumer is driven by a sort of "psychical force" that is exactly like the active force

[8]See, in particular, Chipman (1976, p. 75).

driving a material point in its motion. Yet, to take this analogy seriously means to suppose that the variables involved in the twofold interpretation of the same formal model be logically comparable. But in mechanics forces are vector quantities; further, if a force acting on a material point in $\mathbf{R}^3$ is positional, as indeed it happens to be in most well-known physical phenomena, it can be viewed as a vector field $\mathbf{F}$ from $\mathbf{R}^3$ to $\mathbf{R}^3$, while its component function F_k can be taken to represent the action exerted on the point in the direction of the k-th coordinate axis ($k = 1, 2, 3$). Hence, if the analogy is to be preserved, the "psychical force" acting on a rational consumer must be regarded as a vector quantity; further, as in mechanics a number of important phenomena are governed by positional forces, so in economics it may seem natural to suppose the "psychical forces" involved in the problem of optimal consumer choice to be "positional" as well, the only differences being the following three: first, that in the economic case "position" means a point in commodity space, namely, in the specific situation under discussion, a consumption bundle; second, that a consumption bundle is necessarily non-negative; third, that the commodity space, unlike ordinary space, is not necessarily restricted to three dimensions; as a consequence, the "psychical force" driving a consumer in his rational choices can be viewed as a vector field $\boldsymbol{\varphi}$ from $\mathbf{R}^n_+$ to $\mathbf{R}^n_+$, n being the number of commodities traded in the economy, and its component function φ_k can be taken to represent the "psychical action" exerted on the consumer in the direction of the k-th coordinate axis, i.e., in the direction along which the quantity of the k-th commodity is measured ($k = 1, 2, ..., n$). But then each component function φ_k can be naturally interpreted as the marginal utility function of the k-th commodity[9]; and, consequently, the marginal utility functions of the various commodities become the natural primitives on which to build the theory of optimal consumer choice.

As to total utility, the mechanical analogy is still relevant to understand Pareto's peculiar position. In fact, as we have seen in Section 2, a positional force need not be the gradient of a scalar field; indeed, this is the case only for a special, albeit

[9]From his early writings, Pareto follows Edgeworth (1881) in supposing the marginal utility function of each commodity to generally depend on the quantities consumed of all commodities.

important, class of positional forces, the so-called conservative forces, for which one has $\mathbf{F} = \nabla U$, the scalar field U being called the potential of the conservative force $\mathbf{F}$. Now, given these premises in mechanics, for Pareto it is natural to suppose that, in economics as well, a vector field $\boldsymbol{\varphi}$, representing the "psychical force" driving a consumer in his rational choices, cannot in general be regarded as the gradient of a suitably specified scalar field, so that the marginal utility function φ_k cannot in general be obtained by partially differentiating a total utility function with respect to commodity k ($k = 1, 2, ..., n$). From this perspective, the existence of a total utility function, $\Phi : \mathbf{R}^n_+ \rightarrow \mathbf{R}$, appears to be an exception, rather than the rule; but, of course, when such a function Φ exists, also the "psychical force" $\boldsymbol{\varphi}$ can be regarded as "conservative", so that $\boldsymbol{\varphi} = \nabla\Phi$, i.e., $\varphi_k = \frac{\partial \Phi}{\partial x_k}$, x_k being the quantity consumed of commodity k ($k = 1, 2, ..., n$)[10].

Pareto's efforts to justify his own claim that the vector field $\boldsymbol{\varphi}$ can be viewed as an exact psychical analogue of a physical force $\mathbf{F}$ will prove entirely vain. But, what is worse, such attempts will have negative effects on Pareto's own research activity: for, as we shall presently see, not only will they hinder the development of Pareto's most original contribution in the field of utility theory, namely, his critique of the old "cardinalist" tradition and his related advocay of the new "ordinalist" approach, but they will also mislead him into making serious mistakes. Yet, before discussing these specific issues concerning utility, it is convenient to go back to the general question of the relationship between mechanical and economic equilibrium, as it emerges from the above-quoted passage of the *Cours*.

The basic proposition advanced in that passage entails a contradiction which it is worth stressing. The sort of economy discussed therein is an exchange economy. But since exchange involves at least two agents, it is by definition a social activity; hence an exchange economy cannot but be a social formation, that is, it cannot but consist of a number of agents greater than one. Therefore, if the equilibrium conditions for an exchange economy are to be compared with the equilibrium conditions for a material system, one would naturally expect the material system selected for this

[10]See, in particular, Pareto (1896-97, Vol. I, pp.10-11, fn. (**25**)[1]).

purpose to consist of a number of points greater than one. Yet, if one considers what Pareto calls the "equilibrium equations" of the formal model (namely, equations $[P1]$ and $[P2]$), one immediately realizes that such equations involve only one consumer (respectively, material point) at a time when the economic (respectively, mechanical) interpretation is adopted. However, in a material system consisting of a number of points greater than one, it is hopeless to try to identify a set of "equilibrium conditions" holding for one single point *independently* of the equilibrium conditions holding for *all* the other points belonging to the system: indeed, as we have seen in Section 2, the equilibrium conditions for a discrete material system, as expressed by the Symbolic Equation of Statics (equation (1)), involve all the points belonging to the system in the same overall equation. But then we are forced to conclude that, under the mechanical interpretation, equations $[P1]$ and $[P2]$ can only be regarded as "equilibrium equations" if one is willing to accept that the material system under question is a one-point material system; and indeed, for such a system, equations $[P1]$ and $[P2]$ do express the static equilibrium conditions, as shown at the end of Section 2. This, however, contradicts the requirement that the material system should consist of a number of points greater than one, for only in this case the comparison with an exchange economy might prove to be sensible.

As might be expected, the fallacy emerging under the mechanical interpretation of the formal model is paralleled by a sort of symmetric fallacy, revealing itself under the economic interpretation. As we have just seen, under suitable regularity assumptions, equations $[P1]$ and $[P2]$ do provide a set of conditions that are necessary and sufficient for a mechanical *equilibrium*, provided that the material system under question is a *one-point system*; on the other hand, the same equations $[P1]$ and $[P2]$, when given an economic interpretation, do provide a set of conditions concerning a consumer participating in a pure-exchange competitive economy, which necessarily is a *social* economy, but unfortunately, and contrary to what Pareto's wording seems to suggest, they do *not* represent a set of *equilibrium* conditions for that consumer, let alone for the economy as a whole. As a matter of fact, under suitable regularity *and* concavity assumptions, equations $[P1]$ and $[P2]$ do provide a set of conditions that are necessary and sufficient for determining the consumption bundle optimally chosen by the consumer concerned. But an optimal choice need not be an equilibrium for the obvious reason that, unless the given price system happens to be an equilibrium price system (what cannot certainly be inferred from the specified equations, where prices are taken as arbitrarily fixed parameters), the chosen bundle need not be feasible for the consumer. But feasibility, on which observability depends, appears to be a characteristic feature of any conceivable equilibrium behavior of an economic agent, hence of any possible definition of equilibrium appropriate to the economic science (including, of course, the definition of equilibrium concretely employed in competitive General Equilibrium Theory).

Of course Pareto is perfectly aware that, in order to get a complete set of equilibrium conditions for a pure-exchange competitive economy, the specified equations, providing the conditions for an optimal individual choice (one for each consumer), need to be supplemented by equations of a wholly different nature, whose task is to insure the compatibility of the optimally chosen individual plans at the economy-wide level. In fact, Pareto even coins a revealing expression, "individual economy", to refer to the optimal choice problem of the individual consumer, whose solution is given by equations like $[P1]$ and $[P2]$[11]. Further, when analyzing in detail the general equilibrium problem in a pure-exchange competitive economy, he never forgets integrating the conditions characteristic of the "individual economy" (one set of conditions for each consumer) with the required supplementary conditions, which of course are nothing other than the market-clearing conditions[12]. Yet, all this is apperently forgotten when it comes to comparing the economic equilibrium concept with the mechanical one. But the reason for this forgetfulness is simple: since the epistemological foundations of the two equilibrium concepts are basically different, Pareto, when trying to prove the existence of inexistent similarities between the two concepts, is almost forced to neglect a number of important things concerning economic equilibrium he

[11] See, e.g., Pareto (1892-93, Part I, p. 417) and (1909, App., p. 560).

[12] See, e.g., Pareto (1896-97, Vol. I, pp. 24-26, fn. (**59**)[1]), where equations (1) and (4) correspond to equations $[P1]$ and $[P3]$ above, while equations (5) express the market-clearing conditions appropriate to a pure-exchange economy.

proves to be perfectly aware of when analyzing the latter concept in isolation.

It is convenient to make explicit the basic epistemological differences between the two equilibrium concepts that emerge from the previous discussion. In competitive General Equilibrium Theory, in constructing the equilibrium concept one has to carefully distinguish two levels of the analysis: the "individual" level, at which the individual agents make their rational choices, and the "social" level, at which the issue of the compatibility of the individual choices is taken care of. But no such distinction can possibly apply to the mechanical equilibrium concept: as no rational choice process can obviously be ascribed to a material point, no plausible meaning can be attached to an "individual" level in this context; but, for the same reason, no separate "social" level needs to, or indeed can, be identified here, for no compatibility problem can ever arise in mechanics that is independent of the constraints to which the material points are subject. In this last respect, Pareto is misled into believing that, say, the budget constraints restraining the individual competitive consumers in their optimal consumption choices are exactly like the constraints restraining the motions of material points in a mechanical setting, so that the prices, which are to be taken as fixed parameters in each individual competitive choice problem, can be regarded as the exact economic analogue of the parameters on which the geometrical properties of the constraints appearing in a mechanical equilibrium problem can be supposed to depend[13]. But this analogy is not simply misleading: it is false. As a matter of fact, in mechanics there is no difficulty in supposing that the points belonging to a material

[13]As a matter of fact, as far as the economic side of the comparison is concerned, Pareto does not restrict the analogy to the competitive case, that is, to the case in which prices are taken as fixed parameters at the "individual" level; in his opinion, in fact, the analogy extends also to a more general situation, often referred to as the case of "non-constant" or "variable prices", in which prices are no longer taken as fixed parameters by the individual traders in deciding the global amounts of commodities to trade, but are allowed to vary with the "successive portions"of the commodities that are being traded (see, e.g., Pareto (1892-93, Part I, p. 414) and (1909, App., p. 566 ff.)). This alleged extension of Pareto's argument beyond the strictly competitive case, though a complete failure from a theoretical point of view, is very interesting from a different perspective, for it provides a further demonstration of how dangerously misleading the mechanical analogy can prove in affecting Pareto's thought: in fact, Pareto's efforts in this direction are obviously motivated by his hope to arrive at a formal representation of the constraints to which the economic agents are subject that is closer to the mechanical representation than the one that is allowed for by the purely competitive case. In spite of its interest, however, we shall not discuss this extension here, for this would raise a number of difficult questions that there is no space to deal with properly in this paper.

system be subject to constraints whose geometrical properties depend on the values taken by suitably specified parameters. But it would be wholly meaningless in this context to require that such parameters take on some special "equilibrium values" to be determined as a part of the solution of the very problem under question: for, if the problem is statical, then the constraints, hence the parameters, must necessarily be taken as fixed; if instead the problem is dynamical, then the parameters can indeed be a function of time, but in such a case they must be supposed to depend on time according to some exogenously specified law. In either case, then, it would make no sense to suppose the parameters to take some *endogenously* determined values. But what would be meaningless to assume in mechanics is precisely what has to be assumed in economics: here, in fact, the competitive equilibrium problem can only be solved if the prices (i.e., the parameters of the economic problem at the "individual" level) are allowed to take some endogenously determined "equilibrium values".

For the reasons explained above, Pareto is led to ignore, or even to conceal, the fundamental differences between the equilibrium concepts respectively employed in mechanics and economics. This dangerous attitude, however, causes him to make a number of theoretical mistakes, of which we want now to discuss just two instances.

To simplify the discussion of the first question, let us assume that there exists a scalar field $\Phi : \mathbf{R}^3 \to \mathbf{R}$ such that $\boldsymbol{\varphi} = \nabla\Phi$. As we have seen, under the mechanical interpretation, this means that the active force $\boldsymbol{\varphi}$, being the gradient of the potential Φ, is conservative; under the economic interpretation, that there exists a total ophelimity function Φ, of which the marginal ophelimity functions φ_k are the partial derivatives ($k = a, b, c$). Let $\mathbf{x}^*$ be a value of $\mathbf{x}$ satisfying $[P1]$ and $[P2]$. Under the present assumption, this implies $\delta\Phi(\mathbf{x}^*) = 0$, so that a constrained stationarity of Φ occurs at $\mathbf{x}^*$. Now, when the mechanical interpretation is adopted, $\mathbf{x}^*$ is a mechanical equilibrium independently of $\Phi(\mathbf{x}^*)$ being a local constrained maximum, a local constrained minimum, or neither of them; in fact, the exact characteristics of the critical point only matter for the stability of the equilibrium, an equilibrium being stable (in a specified sense) only if the potential Φ is a maximum, so that $-\Phi$, the potential energy, is a minimum. But when the economic interpretation is adopted, this is no longer the case: in fact, $\mathbf{x}^*$ is an optimal consumption choice (not an "equilibrium", properly speaking) only if $\Phi(\mathbf{x}^*)$ is a constrained maximum of the total ophelimity function in $B(\mathbf{p}, h)$, and a global one, for that matter; hence, in this case, the exact characteristics of the critical point do matter for the very definition of the optimal choice, rather than its stability (whatever this might mean). All this is quite trivial for everybody who is aware of the basic differences existing between the mechanical and the economic equilibrium concept. But Pareto, blinded with the desire to pursue the mechanical analogy to its extreme consequences, makes the unbelievable mistake of qualifying as an "unstable equilibrium" a consumption bundle for which the total ophelimity is a minimum[14].

Finally, let us turn to the last question we want to discuss in this Section. Up to now, following Pareto, we have taken for granted that equations $[P1]$ and $[P2]$ can be indifferently given a mechanical and an economic interpretation. But, as far as the vector equation $[P2]$ is concerned, this is not exactly true. As a matter of fact, no difficulty arises concerning the mechanical interpretation of $[P2]$, which is in fact entirely standard; in particular, as Pareto suggests, the scalar T can be given a well-defined physical meaning, for it can be interpreted as the modulus of the force of reaction of the constraint ("la résistance du plan"), which of course equals (at an equilibrium) the modulus of the active force, $\|\boldsymbol{\varphi}\|$. But when it comes to the economic interpretation of equation $[P2]$, what sort of meaning can be ascribed to the scalar T? In effect, there is no obvious answer to this question.

More fundamentally, the problem appears to be related to the dubious analogy, that Pareto strives to establish, between the components φ_k of the active force $\boldsymbol{\varphi}$, on the mechanical side, and the marginal ophelimities, equally denoted φ_k, on the economic side. In effect, while in rational mechanics there is no possible doubt that the components φ_k of the active force $\boldsymbol{\varphi}$ are scalar quantities, so that the active force $\boldsymbol{\varphi}$ is a vector quantity, in economics, instead, it is far from clear that the marginal

[14]See Pareto (1909, p. 183): "Le point de tangence pourrait aussi être le point le plus bas du sentier, et en ce point l'equilibre serait instable". To appreciate the strength and persistence of the mechanical illusion in modern economics, it should be noted that a similar blunder can be found in a famous article written almost three decades later by Georgescu-Roegen (1936).

ophelimities φ_k can be legitimately taken as scalar quantities, so that it is more than doubtful that the "psychical force" $\boldsymbol{\varphi}$ can be meaningfully regarded as a vector quantity[15] $(k = a, b, c)$. As mentioned above, Pareto will not ebryonically develop his "ordinalist" approach to utility theory before 1898. But in 1896-97, when writing the passage we are commenting upon, he is already perfectly aware that, for the purposes of the determination of the statical "equilibrium" conditions in the consumer's choice problem, it is wholly unnecessary to think of the "psychical force" $\boldsymbol{\varphi}$ as a vector quantity: in fact, as equations $[P3]$ unambiguously reveal, all that matters in $\boldsymbol{\varphi}$ is its direction, which must coincide, at an equilibrium, with the direction of the price vector $\mathbf{p}$; but, contrary to what is suggested by equations $[P2]$, nothing needs to be said or known about the modulus of $\boldsymbol{\varphi}$, which consequently need not be regarded as a vector quantity. Yet, while recognizing that all that matters for the statical "equilibrium" problem is the proportionality of $\boldsymbol{\varphi}$ and $\mathbf{p}$, so that only equations $[P3]$, together with the budget constraint $[P1]$, ought to be considered for this purpose, still Pareto tries to preserve the mechanical analogy by formally deriving equations $[P3]$ from equations $[P2]$, what is wholly incongrous from a mathematical point of view.

Text

4 Dynamics and d'Alembert's Principle in mechanics and economics

For the reasons explained in the Introduction, from the very beginning of his research activity in economics through the whole of his life, Pareto clings to the idea that, in order to fully exploit the mechanical analogy in the field of economic dynamics, one has to follow a roundabout route, hinging on the use of a suitable economic analogue

[15] Here the term "quantity" - scalar or vector quantity, as the case may be - is used in the same sense as in the natural sciences, where it is intended to mean what is usually called by contemporary economists a "cardinally measurable magnitude". On the contrary, what contemporary economists often call an "ordinal variable" is called a "quality" by the natural scientists; correspondingly, the "ordinal measure" of the former is generally referred to as an "index function" by the latter. In this respect, from 1900 onwards, Pareto will adopt the terminology characteristic of the natural sciences.

of d'Alembert's Principle in mechanics[16]. This idea is most clearly stated in the following passage of the *Cours*:

> **586.** [....] En mecanique, le principe de d'Alembert nous permet d'étudier, d'une manière complète, l'état dynamique d'un système. Nous ne faisons encore, en Economie politique, qu'entrevoir un principe analogue[1]. [Pareto's argument continues in the following footnote]
>
> (**586**)[1] Soit, comme d'habitude, Φ l'ophélimité totale. Pour simplifier, considérons trois biens économiques seulment; r_a, r_b, r_c seront les quantités consommées de ces biens, quand il s'agira d'un système économique, et les trois coordonnées d'un point matériel de masse m, quand il s'agira d'un système matériel. Au reste, ce que nous disons s'étendra facilement à plus de trois biens économiques et à l'espace à plus de trois dimensions. Si la consommation de A augmente de dr_a, l'individu fait un gain d'ophélimité exprimé par
>
> $$\varphi_a dr_a \ .$$
>
> L'individu aura donc une tendence à continuer dans la voie qui lui a procuré cette augmentation de bien-être. Supposons qu'on puisse mesurer cette *tendance*, et indiquons-la par
>
> $$\frac{\partial \xi_a}{\partial r_a} dr_a \ ,$$
>
> nous aurons
>
> $$\varphi_a dr_a - \frac{\partial \xi_a}{\partial r_a} dr_a = 0 \ .$$
>
> C'est-à-dire que le gain d'ophélimité sera dépensé pour produire cette tendance.

[16] References to d'Alembert's Principle.
Sensini

Maintenant, considérons le cas général où les quantités varient de δr_a, δr_b, ..., nous aurons

$$(1) \qquad \left(\varphi_a - \frac{\partial \xi_a}{\partial r_a}\right) \delta r_a + \left(\varphi_b - \frac{\partial \xi_b}{\partial r_b}\right) \delta r_b + \cdots = 0 \quad ; \qquad [P4]$$

δr_a, δr_b, ... sont les mouvements virtuels compatibles avec les *liaisons* du système. C'est en particularisant ces liaisons que nous avons l'équation (1) (**59**[1]), les équations (3) (**385**[2]), etc.

Quand il s'agit d'un système materiel, l'équation (1)$[P4]$ n'est autre que celle que donne le principe des mouvements virtuels combiné avec le principe de d'Alembert. Mais quand il s'agit d'un système économique, nous nous trouvons arrêtés, parce que nous ignorons, non seulment la valeur, mais même la nature des fonctions

$$(2) \qquad \frac{\partial \xi_a}{\partial r_a} dr_a \ , \quad \frac{\partial \xi_a}{\partial r_a} dr_a \quad , \ldots \qquad [P5]$$

Au contraire, pour un point matériel, nous pouvons considérer φ_a, φ_b, φ_c comme les forces qui le sollicitent et, alors, les fonctions (2)$[P5]$, prises avec le signe moins, sont les forces d'inertie, et l'on a

$$(3) \qquad \frac{\partial \xi_a}{\partial r_a} = m \frac{d^2 r_a}{dt^2} \ , \qquad \frac{\partial \xi_b}{\partial r_b} = m \frac{d^2 r_b}{dt^2} \ , \ldots \qquad [P6]$$

dt étant le temps pendant lequel le point matériel parcourt la ligne dont les composantes sont dr_a, dr_b, dr_c. Ce sont de semblables équations qu'il nous faudrait pouvoir découvrir pour un système économique.

L'ophélimité totale, quand elle existe (**25**[1]), correspond à la *fonction des forces* en mécanique. C'est-à-dire, c'est la fonction dont les dérivés partielles φ_a, φ_b, ... représentent les forces qui sollicitent le point matériel. Indiquons par $\sum$ une somme qui s'étend à tout le système de points matériels, ou d'individus, et posons

$$(4) \qquad J = -\sum \Phi$$

> La *fonction des forces* est $-J$, et J est ce que, dans la théorie mécanique de la chaleur, l'on appelle l'*énergie potentielle* ou l'*ergal.*[17]

As can be seen, the formal structure of Pareto's argument in the new dynamic context is quite similar to the one he had already experimented with in the static context: after introducing a common formalism, Pareto provides a twofold interpretation of the symbols and the equations, alternatively in terms of mechanical and economic concepts; this, in turn, should convince the reader of the substantive similarity between mechanics and economics in the dynamical field as well.

As already noted in the statical case, also here it should be pointed out that the pairwise comparison suggested by Pareto is far from satisfactory: in fact, while on the economic side of the pair we find a consumer, who is a member of a *social economy* consisting of many agents, on the mechanical side we find instead a *one-point material system* (given equation $[P6]$, there is only one interpretation of equation $[P4]$ that is consistent with the findings of rational mechanics, and such interpretation leaves no doubt as to the nature of the system under discussion). However, as we have already discussed this issue at length in the previous Section, we shall not dwell upon it here.

[17] Pareto (1896-97, Vol. II, pp. 9-11, fn. (**586**)[1]) . To avoid confusions, everywhere in this quotation we have substituted ξ for x: the latter symbol is in fact generally reserved by Pareto to denote consumption (or consumer demand), so that its use in this passage to denote an altogether different function (whose nature will be discussed presently) is really misleading. A first possible negative effect of Pareto's sloppiness in the use of notation is that, having already employed x for a different purpose, he can no longer use it for denoting consumption (or demand); and this may explain why the symbol r_k, generally employed to denote the excess demand for the k-th commodity, is here incongrously interpreted as denoting the amount consumed (or demanded) of that commodity ($k = a, b, c$). However, since the confusion between consumption and excess demand occasionally revealed in this passage may also be the symptom of a more basic conceptual difficulty, on which we shall return later, we have preferred not to correct Pareto's notation in this case.(Anyhow, since $r_k \equiv x_k - q_k$, as long as q_k is taken to represent the fixed endowment of the k-th commodity, we have $dr_k = dx_k$ (or $\delta r_k = \delta x_k$), so that any confusion between consumption and excess demand is irrelevant in the present context.) In the passage quoted in the text one can find a few cross-references to other Sections of the *Cours*: specifically, equation (1) (**59**)[1] and equations (3) (**385**)[2] are different forms of budget equations, whereas footnote (**25**)[1] is the place where the problem of the existence of a total ophelimity function is extensively discussed.

Now, if the idea is accepted that the material system considered in the above-quoted passage is a simple one-point system, then the mechanical interpretation of the formalism appears to be entirely standard. In fact, by substituting equation $[P6]$ into equation $[P4]$, and then replacing the k-th virtual displacement δr_k by the k-th virtual velocity $v_k' = \frac{\delta r_k}{\delta t}$ in $[P4]$ $(k = a, b, c)$, we get the particular version of the Symbolic Equation of Dynamics that is relevant for such a system. But we know from Section 2 that, by virtue of d'Alembert's Principle, the Symbolic Equation of Dynamics (equation (12)) can be obtained from the Symbolic Equation of Statics (equation (1) or, equivalently, equation (3)), so that Pareto's statements are fully justified as far as the mechanical interpretation of the formal model is concerned.

As far as the economic interpretation is concerned, however, the situation immediately appears to be much more problematic. Pareto himself proves to be aware, at least in some degree, of the difficulties to be encountered in this field. In his opinion, the main problem can be stated in the following terms: while in mechanics the meaning of d'Alembert's Principle is perfectly clear and the functions $\frac{\partial \xi_k}{\partial r_k}$ can be given a well-defined physical interpretation, in economics we only have a vague idea of what might be the analogue of d'Alembert's Principle and, in particular, we know very little about the nature of the functions $\frac{\partial \xi_k}{\partial r_k}$[18]. But, in reality, the problem stressed by Pareto is certainly not the most relevant to be faced in this connection: in fact, when trying to develop the theory of economic dynamics along the lines suggested above, well before reaching the relatively minor issue of the economic interpretation of the functions $\frac{\partial \xi_k}{\partial r_k}$, one would encounter, at a much more fundamental level, the preliminary issue of the treatment of time in economic analysis.

The question can be put as follows. Under the mechanical interpretation of the formal model contained in equations $[P6]$, equations $[P4]$ describe the motion of a material point, that is, they give its position as a function of time. The following question then naturally arises: When we try to provide an economic interpretation of

[18]Pareto literally says that "we ignore not only the value, but also the nature of [such] functions". But, as we shall see presently, what he really thinks in the period when he is writing the *Cours* is that, after all, we do know something (albeit very little) about the nature of such functions.

the same formal model, can we find anything related to the behavior of an economic agent that, exactly like the position in space of a material point, can be plausibly viewed as a function of time and, as a consequence, can be supposed to describe the "motion" of that agent in a suitable "economic space"? Only if this question is answered in the affirmative can Pareto's research programme on economic dynamics have any chance of success. So it is crucial to examine Pareto's attempts to find a suitable economic counterpart of the mechanical concepts of position and motion of a material point.

To this end, let us first recall the essential features of the statical choice problem of an optimal consumption bundle by a competitive consumer; in fact, this simple statical problem, already discussed in the previous Section, represents the effective starting point for Pareto's reflexions in the present dynamical context as well. Then, given the price system, $\mathbf{p} \in \mathbf{R}^n_{++}$, and the consumer's endowment, $\mathbf{q} \in \mathbf{R}^n_{+}$, let $\mathbf{x} \equiv \mathbf{x}(\mathbf{p},\mathbf{q}) \in \mathbf{R}^n_{+}$ and $\mathbf{r} \equiv \mathbf{r}(\mathbf{p},\mathbf{q}) \equiv \mathbf{x}(\mathbf{p},\mathbf{q}) - \mathbf{q} \in \mathbf{R}^n$ be the consumer's demand (consumption) and excess demand function, respectively. In the statical context, it is natural to interpret $\mathbf{q}$ and $\mathbf{x}(\mathbf{p},\mathbf{q})$ as the consumer's "initial" and "final position" in the "commodity space", respectively; $\mathbf{r}$ can then be interpreted as the consumer's "displacement". Now, starting from these premises, Pareto tries to dynamize the picture, by first introducing a time parameter and then reinterpreting the consumer's behavioral variables as functions of such parameter.

The most obvious way to pursue this aim, the way that Pareto actually tries to follow, is to suppose that, starting from the "initial position", the consumer will travel through the "commodity space", "moving" towards the "final position" along a "path" that lies in the budget set[19]. By means of this supposition, the theorist can reinterpret the consumer's "position" in the "commodity space", presumably represented by his consumption bundle, as a continuous function of time, so that, as desired, the dynamic behavior of a consumer can be assimilated to the motion of a material point. Yet, this approach, far from solving Pareto's problem, raises a host of formidable conceptual difficulties that it is worth discussing.

[19] "Motion" ("mouvement réel"), "path" ("sentier"), "final position" ("point terminal"), etc.

The first point to stress is that, contrary to Pareto's presumable intentions, the suggested dynamic reinterpretation of the consumer's choice problem cannot be simply grafted onto the static model: in fact, the dynamic reinterpretation, were it taken seriously, would give rise to a theory of consumer's behavior that is different from, and inconsistent with, the standard theory based on statical premises. But this is peculiar result; in particular, it is something which is absolutely unknown in mechanics, where it would be inconceivable to have a dynamical theory of a certain phenomenon contradicting the results of the statical theory of the same phenomenon. To show that the proposed dynamical theory of consumer choice would be inconsistent with the standard statical theory is trivial. Suppose that, given $\mathbf{p}$, the "initial position" $\mathbf{q}$ is not an optimal choice for the consumer's preferences; then $\mathbf{r}(\mathbf{p},\mathbf{q}) \equiv \mathbf{x}(\mathbf{p},\mathbf{q}) - \mathbf{q} \neq \mathbf{0}$, so that in the present case the statical theory predicts a non-vanishing, finite "displacement" $\mathbf{r}(\mathbf{p},\mathbf{q})$; moreover, given the mentalistic character of the choice process that is being discussed here, such "displacement" should be thought of as taking place instantaneously, and similarly the "final position" $\mathbf{x}(\mathbf{p},\mathbf{q})$ should be supposed to be "instantaneously reached" by the consumer (this is indeed possible, since such position is only "reached" in his mind). But suppose now that, as required by the suggested dynamical theory, the consumer starts to move from the "initial position" $\mathbf{q}$, his motion in the commodity space being a continuous function of time. Due to the continuity assumption, after a vanishing interval of time he can only be infinitely close to the "initial position"; this means, however, that he will not be where his preferences would dictate, and the statical theory of choice would predict, him to be.

As we have just seen, owing to the mentalistic character of the choice process, certain phenomena (in particular, those finite changes in "position" that might be called "finite instantaneous displacements") can occur in the "commodity space" that would be meaningless in ordinary space. This is enough to conclude that the "commodity space" is not exactly like ordinary space, that the "position" of an economic agent in the first kind of space is not exactly comparable to the position of a material point in the second one, and, more generally, that the relationship between time and space in economics is not exactly the same as in mechanics. We have reached this conclusion by using an individual choice problem, with its peculiar mentalistic properties, as our frame of reference. But this is certainly not the only economic context where the relationship between time and space appears to be inconsistent with that characteristic of rational mechanics. Take for instance the "classical barter problem", so prominent in the early contributions of Jevons, Walras, Edgeworth, etc., where two traders, each endowed with a given amount of one consumption good, are supposed to trade with one another. Then, if the two traders behave competitively, and if a competitive equilibrium price ratio is announced, we can imagine the two traders to "instantaneously" exchange the equilibrium amounts of the two goods, thereby "jumping" from the "initial position" to the "final equilibrium position". Once again we find here an instance of a "finite instantaneous displacement". But this phenomenon cannot be reconciled with the central idea of mechanical dynamics, namely, the idea of motion as a continuous function of time; and this is particularly disturbing for the founders of neoclassiacl economics. In effect, the desire to eliminate such disturbing differences with mechanical dynamics is probably the main reason why Jevons, the first to formally discuss and solve the "classical barter problem", after reviewing the alleged limitations of his own solution, suggests to replace the "finite displacements" and the ordinary equations appearing in it with "infinitesimal displacements" and "differential equations", respectively[20].

Returning now to Pareto, it should be pointed out that he occasionally seems to realize some of the difficulties surrounding the introduction of the time element in both individual choice theory and general equilibrium theory, and consequently to perceive some of the obstacles hampering the construction of a theory of economic dynamics. The most explicit discussion of these issues can be found in a passage appearing towards the end of Vol. II of the *Cours*, within a chapter devoted to the discussion of "economic crises", where Pareto goes back to the problems raised in the passage quoted at the beginning of this Section and strives to provide a few concrete

[20] According to Jevons (1871, p. 138), the suggested replacement would allow one "to have a complete solution of the problem in all its natural complexity", a solution comparable to those offered by mechanical dynamics to its characteristic problems. It goes without saying that Jevons' suggestion will produce no effect.

suggestions (the only ones to be found in all of his writings) concerning a possible economic interpretation of d'Alembert's Principle and, in particular, of the functions $\frac{\partial \xi_k}{\partial r_k}$[21]. First of all, Pareto tries to clarify the difficult issue of the time dimension to be associated to the variables involved in the consumer choice problem, the same problem on which we have already dwelt at length before. In this connection he writes:

> **928.** La considération des crises nous porte en plein dans l'étude de la dynamique des systèmes économiques, et il sera bon que [...] nous tâchions de nous rendre compte des conditions de cet équilibre.
>
> Il nous faut, pour cela, abandonner la considération des consommations isolées et considérer des consommations journalières, mensuelles, annuelles, etc. [...] Nous considérons donc les consommations qui ont lieu, en moyenne, dans l'unité de temps[1]. [Pareto's argument continues in the following footnote]
>
> (**928**)[1]Cela change légèrement la signification des quantitées consommées
>
> $$q_a + r_a, \quad q_a + r_a, \; ...$$
>
> Il faut supposer que ce sont là les quantitées consommées dans l'unité de temps. Φ, φ_a, φ_b, ... se rapportent alors aussi à l'unité de temps. Il en est de même de toutes les autres quantités r_a, r_b, ...[...] Toutes ces quantités deviennent alors des quantités du genre des *vitesses*.

This passage unwillingly reveals a basic difficulty concerning the specification of the time dimension of the economic variables appearing in rational choice and general equilibrium problems. Such difficulty is certainly not peculiar to Pareto, for it manifests itself, under different guises, in practically all neoclassical economists[22]; but

[21]See Pareto (1896-97, Vol. II, pp. 280-284, fn. (**928**)[1] and (**928**)[2]).

[22]To keep to the first generation, both Jevons and Fisher devote a lot of effort to the issue of the time dimension of the analysis, without succeeding, however, in providing a satisfactory solution to it; as a matter of fact, they both fall into serious mistakes concerning this point (see Jevons

Pareto's treatment has the advantage of making the issue more explicit. The question under discussion is still that of dynamizing the consumer's statical choice problem. But here Pareto suddenly seems to realize that, since any dynamic theory presupposes the existence of a time dimension, but the variables appearing in the static problem have *prima facie* no time dimension, no dynamization of the static problem is possible unless the relevant variables are suitably reinterpreted. As a consequence, Pareto suggests that the variables x_k (q_k, r_k), which had been originally interpreted, in the static problem, as either *stocks* (referred to a specified instant) or *flows* (referred to a single specified period of time), be reinterpreted as *rates of flow per unit of time* in the dynamic (or, better, dynamizable) version of the problem[23].A similar reinterpretation indirectly extends to all the other variables of the model, in particular to the total and the marginal ophelimity functions, Φ and φ_k. According to Pareto, by means of

(1871, pp. 117-121) and Fisher (1892, pp. 19-21, 103). A better (though by no means conclusive) treatment is offered by Walras (1954, particularly Lesson 35), who at least manages to keep clear of logical mistakes.

[23]The expressions used by Pareto in order to distinguish the static from the dynamic interpretation of the relevant variables are not wholly rigorous. In particular, when he qualifies the variable $x_k \equiv q_k + r_k$ in the static problem as an "isolated consumption", he certainly wants to rule out the possibility of interpreting it as a "uniformly repeated consumption", that is, as a "rate of flow of consumption per unit of time", but he does not rule out the possibility of interpreting it as either a "stock" or a "flow" (provided that the act of consumption is "isolated", i.e., not repeated). In effect, contrary to what is often held, the real contraposition, as far as the time dimension is concerned, is between a "rate of flow", on one side, and either a "stock" or a "flow", on the other: for while a "rate of flow" has a time dimension (its dimensions being FT^{-1}, where F represents the dimension intrinsic to the "flow" and T the time dimension), neither a "stock" nor a "flow", *per se*, has a time dimension. Now, for the discussion's sake, let us take for granted what Pareto appears to suggest, but is otherwise questionable, namely, let us suppose that the distinctive feature of a variable involved in a static problem is that it lacks any time dimension; then both a "stock" and a "flow", but not a "rate of flow", would satisfy the stated condition. From this point of view, therefore, the fact that Pareto leaves unspecified whether the variable x_k, appearing in the consumer's statical choice problem, is to be interpreted as a "stock" or as a "flow", is after all a minor ambiguity. Moreover, it is an ambiguity that is, so to speak, intrinsic to the problem under discussion. For, as we know, x_k can be viewed as either the consumer's demand for or the consumer's consumption of the k-th commodity. But when x_k is viewed in the first way, it may seem natural to interpret it as a "stock": for then x_k can be seen as the sum of the consumer's endowment, q_k, and of his excess demand, r_k; and while q_k, being the amount of the commodity held by the consumer at a certain instant of time, is naturally interpreted as a stock, r_k, being the amount that the consumer actually plans to trade, can be easily referred to the instant at which the trade is supposed to take place, and this instant can in turn be identified with the one to which q_k is referred). On the contrary, when x_k is viewed in the second way, it may seem natural to interpret it as a "flow" (for the consumption x_k can be plausibly supposed to be planned by the consumer with a finite time period in mind). In conclusion, even in so simple a problem as the present one, the relationship between the economic variables and time appears to be exceedingly complex. The reasons for this will be discussed presently in the text.

this reinterpretation the relevant economic variables would acquire a nature similar to the nature of velocities in mechanics, the obvious implication being that, as velocities are instrumental in the development of mechanical dynamics, so the reinterpreted economic variables will be instrumental in the development of economic dynamics. But this hope is groundless.

Leaving temporarily aside the absurd idea, on which we shall come back presently, that a rate of flow of consumption can share any of the analytical properties of a velocity, let us focus attention on Pareto's central proposition that in economics there exist some variables that are susceptible of a twofold interpretation, either statical or dynamical, so that one can switch from statics to dynamics simply by changing the interpretation of such variables. Now, contrary to what Pareto appears to believe, nothing similar can be found in mechanics. Here, it is true, there are *equations* or *relations* that are susceptible of a twofold interpretation and that, according to the interpretation chosen, can be employed to derive either statical or dynamical results (as we have seen, d'Alembert's Principle exploits precisely this fact). But it would be meaningless to assert that a position or a displacement, a velocity or an acceleration, a force or a work or a power change their respective natures according to whether we take a static or a dynamic point of view; a position remains a position from whatever point of view we look at it, and the same holds for the other quantities. Of course, statics can be developed without mentioning time or quantities depending on time, such as velocity or acceleration. But this does not mean that such quantities cease to exist or change their nature in a statical context; and in effect, if one so desires, one can legitimately say that velocity is constant or acceleration nil at a static equilibrium.

> Il faut tenir compte de toutes ces circonstances pour avoir les conditions de l'équilibre dynamique du système économique[2].
>
> (**928**)[2] Si, en maintenant les notations de la note précédente, nous prenons une unité de temps assez petite, par exemple le jour, r_a, r_b, ... qui se rapportent à cette unité représenteront les *vitesses* des débits. Nous pourrons confondre les différences finies avec les différences infinitésimales et, en in-

diquant par t le temps, considérer comme égales les expressions

$$\Delta r_a = \frac{dq_a}{dt} \ ,$$

q_a étant la consommation qui a lieu depuis une certaine origine du temps jusqu'au temps t.

Le changement dans la vitesse de la consommation de A sera mesuré par $\frac{dr_a}{dt}$.

Nous pourrons représenter par $\mathsf{f}_a \left(\frac{dr_a}{dt}\right)$ la peine qu'il faut que l'individu se donne pour effectuer ce changement;

f_a étant une certaine fonction, sur laquelle, comme nous l'avons déjà dit (**586**[1]), l'expérience ne nous a malheureusement pas encore donné de renseignements.

Le gain d'ophélimité que fait l'individu, par la consommation δr_a, sera donc

$$(\varphi_a - \mathsf{f}_a)\,\delta r_a$$

et pour l'équilibre on devra avoir

$$(1) \qquad (\varphi_a - \mathsf{f}_a)\,\delta r_a + (\varphi_b - \mathsf{f}_b)\,\delta r_b + \cdots = 0 \ . \qquad [P6]$$

On a, en autre,

$$(2) \qquad r_a p_a + r_b p_b + \cdots + r_e = r_s p_s + r_t p_t + \cdots \qquad [P7]$$

S, T ... étant les capitaux que possède l'individu, et r_e la quantité de numéraire qu'il ajoute à son épargne, ou qu'il prélève sur celle-ci.

[...]

Les équations (3), (5), (6), (7) de (**100**[1]) subsistent toujours, et, avec les équations (1)[P6], (2)[P7], [...] que nous venons de trouver, elles nous donnent les équations générales de la dynamique des systèmes économiques.

L'équation (2)[$P7$] donne

$$p_a \delta r_a + p_b \delta r_b = 0 \ , \qquad p_a \delta r_a + p_c \delta r_c = 0 \ , \ \ldots$$

ce qui transforme l'équation (1)[$P5$] dans les équations suivantes

$$(1bis) \qquad \frac{1}{p_a}\left(\varphi_a - \mathsf{f}_a\right) = \frac{1}{p_a}\left(\varphi_a - \mathsf{f}_a\right) = \frac{1}{p_a}\left(\varphi_a - \mathsf{f}_a\right) = \cdots \qquad [P8]$$

Text